Chapter 12: Introduction to the Laplace Transform

12.1 Definition of the Laplace Transform

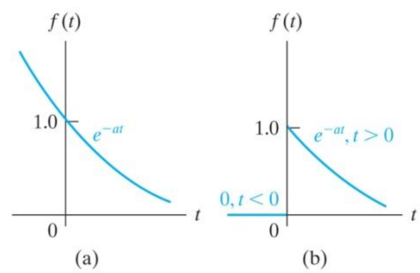
The Laplace transform is used to transform a set of integrodifferential equations from the *time* domain to a set of algebraic equations in the *frequency* domain.

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt \qquad \textit{where} \qquad F(s) = \mathcal{L}\{f(t)\}$$

Since we only consider for positive values of time it is often referred to as the **one-sided** or **unilateral Laplace transform.**

Since f(t) may have a discontinuity at the origin we use 0^- and 0^+ to refer to the values just before and just after the discontinuity.

0^- includes the discontinuity
0^+ excludes the discontinuity



We will use 0^- to do our evaluation but ignore the values for $t < 0^-$, since we consider that to be handled by the initial conditions.

Two –types

Functional: the Laplace transform of a specific function (i.e. sinωt or t)

Operational: defines a general mathematical property of the Laplace transform (i.e. f´(t))

12.2 The Step Function

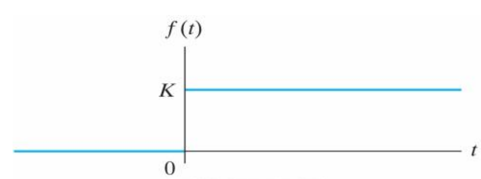
Defined as $Ku(t)$where

$$Ku(t) = 0 \; for \; t < 0$$

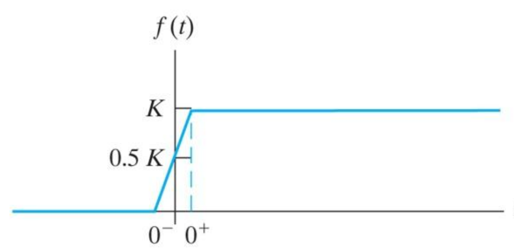
$$Ku(t) = K \; for \; t > 0$$

When K = 1, we have the **unit step**

Figure 12.2 The step function.



It is undefined at t = 0, if we must define assume linearity and $Ku(0) = 0.5K$



For a discontinuity that occurs at a time other than t = 0 for example t = a

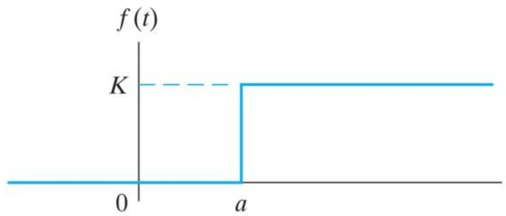
For a > 0

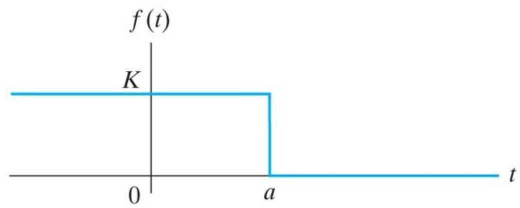
$$Ku(t-a) = 0 \; for \; t < a$$

$$Ku(t-a) = K \; for \; t > 0$$

For a < 0

$$Ku(a-t) = K \; for \; t < a \qquad and \qquad Ku(a-t) = 0 \; for \; t > a$$





Step functions can be used to write a mathematical expression that is non-zero for a finite duration and is defined for all positive values.

Ex. $Ku(t-1) - u(t-3)$ this expression is K for $1 < t < 3$ else 0

Think $u(t-1)$turns "ON" the function to K at t = 1and $u(t-3)$turns "OFF" the function to K at t = 3.

Review example 12.1

12.3 The Impulse Function

Allows for the definition of the derivative at a discontinuity.

An *impulse* is a signal with infinite amplitude and zero duration.

For the derivative of a function at a discontinuity assume the function varies linearly across the discontinuity.

For the function in figure 12.8 as ϵ approaches zero, (where ϵ is a variable value parameter)

The derivate between ±ϵ is constant at $\frac{1}{2}\epsilon$

And $-ae^{-a(t-\epsilon)}$ for t > ϵ as shown in Figure 12.9

As $\epsilon \to 0$; between ±ϵ

The value of $f'(t)$becomes infinite

The area under $f'(t)$ is constant (unity here)

The duration approaches zero

Unit impulse function; denoted $\delta(t)$ (aka Dirac delta function)

Figure 12.8 A magnified view of the discontinuity in Fig. 12.1(b), assuming a linear transition between −ϵ and + ϵ.

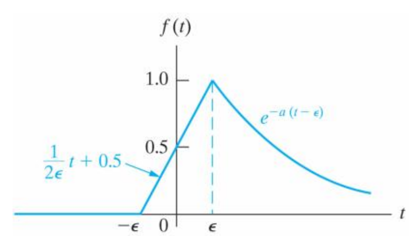
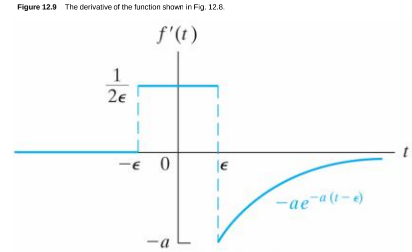


Figure 12.9 The derivative of the function shown in Fig. 12.8.



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If the area under the curve is other than unity, the function is denoted $K\delta(t)$ where K is the area and is referred to as the strength of the impulse function.

Review

An impulse function is created from a variable-parameter function whose parameter approaches zero; the variable-parameter function has the following characteristics:

- The amplitude approaches infinity
- The duration of the function approaches zero
- The area under the function is constant as the parameter changes

Mathematical definition of an impulse function:

$$\int_{-\infty}^{\infty} K\delta(t)dt = K \quad and \quad \delta(t) = 0\, for\; t \neq 0$$

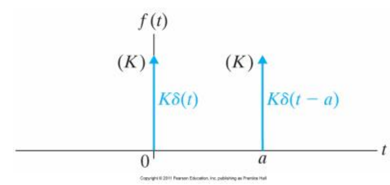
The area under the impulse function is constant and the function is zero everywhere but at t = 0.

For an impulse at t = a; the impulse is written

$$K\delta(t-a)$$

Graphically the symbol of an impulse function is an arrow with its strength (K) next to it.

Figure 12.11 A graphic representation of the impulse $K\delta(t)$ and $K\delta(t-a)$.



Sifting property

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$$

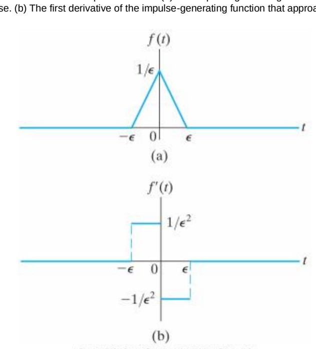
Thus the impulse function sifts out everything except the value of the function at t = a. This is valid since $\delta(t-a)$ is zero everywhere but t = a.

Using the sifting property to find the Laplace transform of the impulse function…

$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)e^{-st}dt = \int_{0^-}^{\infty} \delta(t)dt = 1$$

Finding the Laplace of the derivative of the impulse function; given the derivate from figure.

Figure 12.12 The first derivative of the impulse function. (a) The impulse-generating function used to define the first derivative of the impulse. (b) The first derivative of the impulse-generating function that approaches $\delta(t)$ as $\epsilon \to 0$.



$$\mathcal{L}\{\delta'(t)\} = \lim_{\epsilon\to 0}\left[\int_{-\epsilon}^{0^-} \frac{1}{\epsilon^2}e^{-st}dt + \int_{0^+}^{\epsilon}\left(-\frac{1}{\epsilon^2}\right)e^{-st}dt\right]$$

$$= \lim_{\epsilon\to 0}\left[\frac{e^{s\epsilon}+e^{-s\epsilon}-2}{s\epsilon^2}\right] = \lim_{\epsilon\to 0}\left[\frac{se^{s\epsilon}-se^{-s\epsilon}}{2s\epsilon}\right] = \lim_{\epsilon\to 0}\left[\frac{s^2e^{s\epsilon}+s^2e^{-s\epsilon}}{2s}\right] = s$$

General form for the nth derivative

$$\mathcal{L}\{\delta^{(n)}(t)\} = s^n$$

Another way of expressing the impulse

$$\delta(t) = \frac{du(t)}{dt}$$

12.4 Functional Transforms

The Laplace transform of a special function of t

$$i.)\ \mathcal{L}\{u(t)\} = \int_{0^-}^{\infty} f(t)e^{-st}dt = \int_{0^+}^{\infty} 1e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_{0}^{\infty} = \frac{1}{s}$$

$$ii.)\ \mathcal{L}\{e^{-at}\} = \int_{0^-}^{\infty} e^{-at}e^{-st}dt = \int_{0^+}^{\infty} e^{-(a+s)t}dt = \frac{1}{s+a}$$

$$iii.)\ \mathcal{L}\{\sin\omega t\} = \int_{0^-}^{\infty} \frac{e^{j\omega t} - e^{-j\omega t}}{2j} e^{-st}dt = \int_{0^+}^{\infty} \frac{e^{(s-j\omega)t} - e^{-(s+j\omega)t}}{2j} dt$$

$$= \frac{1}{2j}\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right) = \frac{\omega}{s^2+\omega^2}$$

TABLE 12.1 An Abbreviated List of Laplace Transform Pairs

Type	$f(t)\ (t > 0-)$	$F(s)$
(impulse)	$\delta(t)$	1
(step)	$u(t)$	$\frac{1}{s}$
(ramp)	t	$\frac{1}{s^2}$
(exponential)	e^{-at}	$\frac{1}{s+a}$
(sine)	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
(cosine)	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
(damped ramp)	te^{-at}	$\frac{1}{(s+a)^2}$
(damped sine)	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
(damped cosine)	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

12.5 Operational Transforms

How mathematical operations performed on f(t) or F(s) are converted into the opposite domain.

Multiplication by a Constant

$$If \quad \mathcal{L}\{f(t)\} = F(s) \quad then \quad \mathcal{L}\{kf(t)\} = kF(s)$$

Addition (Subtraction)

$$\mathcal{L}_1\{f_1(t)\} = F_1(s); \qquad \mathcal{L}_2\{f_2(t)\} = F_2(s); \qquad \mathcal{L}_3\{f_3(t)\} = F_3(s)$$

$$\mathcal{L}_1\{f_1(t) + f_2(t) - f_3(t)\} = F_1(s) + F_2(s) - F_3(s)$$

Differentiation

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = \int_{0^-}^{\infty} \left[\frac{df(t)}{dt}\right] \mathrm{e}^{-\mathrm{st}}\mathrm{dt} = sF(s) - f(0^-)$$

*Solved above using integration by parts: $u = e^{-st}$ *and* $dv = \left[{df(t)}/{dt}\right] dt$

$$If \quad g(t) = \frac{df(t)}{dt} \quad then \quad G(s) = sF(s) - f(0^-)$$

$$\mathcal{L}\left\{\frac{dg(t)}{dt}\right\} = \int_{0^-}^{\infty} \left[\frac{d^2f(t)}{dt^2}\right] \mathrm{e}^{-\mathrm{st}}\mathrm{dt} = s^2F(s) - sf(0^-) - \frac{df(0^-)}{dt}$$

General form:

$$\mathcal{L}\left\{\frac{d^nf(t)}{dt^n}\right\} = s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\frac{df(0^-)}{dt} - s^{n-2}\frac{d^2f(0^-)}{dt^2} \dots - \frac{d^{n-1}f(0^-)}{dt^{n-1}}$$

Integration

Integration in the time domain corresponds to division by s in the frequency domain

$$\mathcal{L}\left\{\int_{0^-}^{t} f(x)dx\right\} = \int_{0^-}^{\infty} [f(x)dx]e^{-st}dt$$

Integration by parts

$$u = \int_{0^-}^{t} [f(x)dx] \;\; \text{and} \;\; dv = e^{-st}dt \quad thus \quad du = f(t)dt \;\; and \;\; v = \frac{-e^{-st}}{s}$$

$$\mathcal{L}\left\{\int_{0^-}^{t} f(x)dx\right\} = \frac{-e^{-st}}{s}\int_{0^-}^{t}(f(x)dx) + \int_{0^-}^{\infty}\frac{e^{-st}}{s}f(t)dt = \frac{F(s)}{s}$$

Translation (time domain)

Translation in the time domain is the same as multiplication by an exponent in the frequency domain.

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s) \;\; for\; a > 0$$

Thus given:

$$\mathcal{L}\{tu(t))\} = \frac{1}{s^2} \quad then \quad \mathcal{L}\{(t-a)u(t-a)\} = \frac{e^{-as}}{s^2}$$

Proof:

$$\mathcal{L}\{f(t-a)u(t-a)\} = \int_{0^-}^{\infty} u(t-a)f(t-a)\mathrm{e}^{-\mathrm{st}}\mathrm{dt} = \int_{a}^{\infty} f(t-a)\mathrm{e}^{-\mathrm{st}}\mathrm{dt}$$

The limits change since the unit step = 1 for t > a and zero everywhere else.

Changing the variable of integration

$$x = t - a; x = 0 \; when \; t = 0 \; and \; x = \infty \; for \; t = \infty \;\; dx = dt$$

$$\mathcal{L}\{f(t-a)u(t-a)\} = \int_{0^-}^{\infty} f(x)e^{-s(x+a)}dx = e^{-as}\int_{0^-}^{\infty} f(x)e^{-sx)}dx = e^{-as}F(s)$$

Translation (frequency domain)

Translation in the frequency domain is the same as multiplication by an exponent in the time domain.

$$\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$$

Ex.

$$\mathcal{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}\{e^{-at}\cos\omega t\} = \frac{s+a}{(s+a)^2+\omega^2}$$

Scale Changing

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right); \; a > 0$$

Ex.

$$\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$$

$$\mathcal{L}\{\cos\omega t\} = \frac{1}{\omega}\frac{\frac{s}{\omega}}{\left(\frac{s}{\omega}\right)^2+1} = \frac{s}{s^2+\omega^2}$$

TABLE 12.2 An Abbreviated List of Operational Transforms

Operation	$f(t)$	$F(s)$
Multiplication by a constant	$Kf(t)$	$KF(s)$
Addition/subtraction	$f_1(t) + f_2(t) - f_3(t) + \cdots$	$F_1(s) + F_2(s) - F_3(s) + \cdots$
First derivative (time)	$\frac{df(t)}{dt}$	$sF(s) - f(0^-)$
Second derivative (time)	$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0^-) - \frac{df(0^-)}{dt}$
nth derivative (time)	$\frac{d^nf(t)}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\frac{df(0^-)}{dt} - s^{n-3}\frac{df^2(0^-)}{dt^2} - \cdots - \frac{d^{n-1}f(0^-)}{dt^{n-1}}$

TABLE 12.2 An Abbreviated List of Operational Transforms

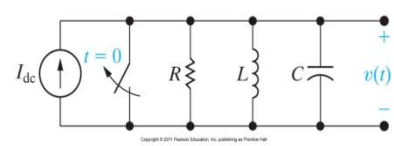
Operation	$f(t)$	$F(s)$
Time integral	$\int_0^t f(x)\,dx$	$\frac{F(s)}{s}$
Translation in time	$f(t-a)u(t-a),\ a>0$	$e^{-as}F(s)$
Translation in frequency	$e^{-at}f(t)$	$F(s+a)$
Scale changing	$f(at),\ a>0$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
First derivative (s)	$tf(t)$	$-\frac{dF(s)}{ds}$
nth derivative (s)	$t^n f(t)$	$(-1)^n\frac{d^nF(s)}{ds^n}$
s integral	$\frac{f(t)}{t}$	$\int_s^\infty F(u)\,du$

12.6 Applying the Laplace Transform

The Laplace transform can be used to solve an ordinary integrodifferential equation.

The equation at the node for the RLC circuit:

$$\frac{v(t)}{R}+\frac{1}{L}\int_0^t v(x)dx + C\frac{dv(t)}{dt} = I_{dc}u(t)$$



Transforming to s-domain

$$\frac{V(s)}{R}+\frac{1}{L}\frac{V(s)}{s}+C[sV(s)-v(0^-)] = I_{dc}\frac{1}{s}$$

Solving for the unknown:

$$V(s)\left(\frac{1}{R}+\frac{1}{sL}+sC\right)=\frac{I_{dc}}{s} \quad\rightarrow\quad V(s)=\frac{\frac{I_{dc}}{s}}{s^2+\frac{1}{RC}s+\frac{1}{LC}}$$

Then return the function to the time domain

$$v(t)=\mathcal{L}^{-1}\{V(s)\}$$

The equation will need to be checked for validity.

Note: For clarity the notation is normally simplified by removing the parenthetical references i.e.

$$\mathcal{L}\{v\} = V \text{ or } v=\mathcal{L}^{-1}\{V\} \qquad \mathcal{L}\{f\}=F \text{ or } fv=\mathcal{L}^{-1}\{F\}$$

12.7 Inverse Transforms

Rational function: one that can be expressed in the form of a ratio of two polynomials in s such that no nonintegral powers of s appear in the polynomials

General form:

$$F(s)=\frac{N(s)}{D(s)}=\frac{a_ns^n+a_{n-1}s^{n-1}+\cdots a_1s^1+a_0}{b_ms^m+b_{m-1}s^{m-1}+\cdots b_1s^1+b_0}$$

Where *a* and *b* are real constants; and *m* and *n* are positive integers

Proper rational function: if $m > n$

Improper rational function: if $m \leq n$

Only a proper rational function can be expanded as a sum of partial fractions.

Partial Fraction Expansion: Proper Rational Functions

A proper rational function is expanded into a sum of partial fractions by writing a series of terms for each root of D(s).

Example $$\frac{s+6}{s(s+3)(s+1)^2}$$

The denominator has 4 roots; two distinct at $s = 0$ & $s = -3$; one repeated at $s = -1$ Yielding a partial fraction expansion of

$$\frac{s+6}{s(s+3)(s+1)^2} \equiv \frac{K_1}{s} + \frac{K_2}{s+3} + \frac{K_3}{(s+1)^2} + \frac{K_4}{s+1}$$

Now we must find the values for the constants. The method must be developed for the four general forms of an equation (or more accurately, its root type).

1. Real and distinct
2. Complex and distinct
3. Real and repeated
4. Complex and repeated

Note: The partial fraction expansion is identical to the original equation; therefore both sides must be equal for all values of s.

Partial Fraction Expansion: Distinct Real Roots of D(s)

Steps for determining the coefficients

i. Multiple both sides of the expression by a factor equal to the denominator beneath one of the K values
ii. Evaluate both sides at the root corresponding to the multiplied factor
iii. The right side of the equation will give the K being determined while the left its numerical value.
iv. Repeat the above steps for all remaining denominators

Example $$F(s) = \frac{96(s+5)(s+12)}{s(s+8)(s+6)} = \frac{K_1}{s} + \frac{K_2}{s+8} + \frac{K_3}{s+6}$$

Multiple both sides by s then evaluate for $s = 0$

$$\frac{96(s+5)(s+12)}{(s+8)(s+6)} = K_1 + \frac{sK_2}{s+8} + \frac{sK_3}{s+6}$$

$$\frac{96(5)(12)}{8(6)} = K_1 = 120$$

Repeating for *s* + 8 and *s* = -8

$$\frac{96(s+5)(s+12)}{s(s+6)} = \frac{K_1(s+8)}{s} + K_2 + \frac{K_3(s+8)}{s+6}$$

$$\frac{96(-3)(4)}{-8(-2)} = K_2 = -72$$

Again for *s* + 6 and *s* = -6

$$\frac{96(s+5)(s+12)}{s(s+8)} = \frac{K_1(s+6)}{s} + \frac{K_2(s+6)}{s+8} + K_3$$

$$\frac{96(-1)(6)}{-6(2)} = K_3 = 48$$

Therefore

$$F(s) = \frac{96(s+5)(s+12)}{s(s+8)(s+6)} = \frac{120}{s} + \frac{48}{s+6} - \frac{72}{s+8}$$

The validity of this solution needs tested; can pick any value for s (try a root of N(s))

$$Testing\ with\ s = -5: \qquad \frac{120}{-5} + \frac{48}{1} - \frac{72}{3} = -24 + 48 - 24 = 0$$

$$Testing\ with\ s = -12: \qquad \frac{120}{-12} + \frac{48}{-6} - \frac{72}{-4} = -10 - 8 + 18 = 0$$

Since the equation appears valid we would then perform the inverse transform to the time domain

$$\mathcal{L}^{-1}\left\{\frac{96(s+5)(s+12)}{s(s+8)(s+6)}\right\} = (120 + 48e^{-6t} - 72e^{-8t})u(t)$$

Review assessment problem 12.3

Partial Fraction Expansion: Distinct Complex Roots of D(s)

The same steps are used for distinct complex roots as for real roots only there will be complex values.

$$Example: \qquad F(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)}$$

$$Where \qquad s^2 + 6s + 25 = (s+3-j4)(s+3+j4)$$

$$\frac{100(s+3)}{(s+6)(s^2+6s+25)} = \frac{K_1}{s+6} + \frac{K_2}{s+3-j4} + \frac{K_3}{s+3+j4}$$

Solving for the constants

$$K_1 = \frac{100(s+3)}{(s^2+6s+25)} = \frac{100(-3)}{25} = -12$$

$$K_2 = \frac{100(s+3)}{(s+6)(s+3+j4)} = \frac{100(j4)}{(3+j4)(j8)} = 6 - j8 = 10e^{-j53.13}$$

$$K_3 = \frac{100(s+3)}{(s+6)(s+3-j4)} = \frac{100(-j4)}{(3-j4)(-j8)} = 6 + j8 = 10e^{j53.13}$$

Thus

$$\frac{100(s+3)}{(s+6)(s^2+6s+25)} = \frac{-12}{s+6} + \frac{10\angle-53.13°}{s+3-j4} + \frac{10\angle 53.13°}{s+3+j4}$$

Observe that the constants of the complex roots t are conjugated pairs just like the roots.

Again the solution should be tested for validity.

$$For\ s = -3; \quad \frac{-12}{3} + \frac{10\angle-53.13°}{-j4} + \frac{10\angle 53.13°}{j4} = -4 + 2 + j1.5 + 2 - j1.5 = 0$$

Taking the inverse

$$\mathcal{L}^{-1}\left\{\frac{100(s+3)}{(s+6)(s^2+6s+25)}\right\} = \left(-12e^{-6t} + 10e^{-j53.13°}e^{-(3-j4)t} + 10e^{j53.13°}e^{-(3+j4)t}\right)u(t)$$

Simplifying by adding the conjugated pairs

$$10e^{-j53.13°}e^{-(3-j4)t} + 10e^{j53.13°}e^{-(3+j4)t} = 10e^{-3t}\left(e^{j(4t-53.13°)} + e^{-j(4t-53.13°)}\right) = 20e^{-3t}\cos(4t - 53.13°)$$

$$\mathcal{L}^{-1}\left\{\frac{100(s+3)}{(s+6)(s^2+6s+25)}\right\} = (-12e^{-6t} + 20e^{-3t}\cos(4t-53.13°))u(t)$$

Now whenever D(s) contains complex roots we can write the factor form as:

$$\frac{K}{s+\alpha-j\beta} + \frac{K^*}{s+\alpha+j\beta}$$

Where the inverse transform is:

$$\mathcal{L}^{-1}\left\{\frac{K}{s+\alpha-j\beta} + \frac{K^*}{s+\alpha+j\beta}\right\} = 2|K|e^{-\alpha t}\cos(\beta t + \theta)$$

Partial Fraction Expansion: Repeated Real Roots of D(s)

Steps for determining the coefficients

i. Multiple both sides by the root with multiplicity r to the r[th] power.
ii. Evaluate both sides at the root corresponding to the multiplied factor
iii. To find the next constant for the root r-1; differentiate both sides.
iv. Evaluate both sides at the root

v. Repeat steps 3 and 4 until all the repeated roots have been accounted for.

Example: $$F(s) = \frac{100(s+25)}{s(s+5)^3} = \frac{K_1}{s} + \frac{K_2}{(s+5)^3} + \frac{K_3}{(s+5)^2} + \frac{K_4}{s+5}$$

Multiplying by s and solving for s = 0;

$$K_1 = \frac{100(s+25)}{(s+5)^3} = \frac{100(25)}{5^3} = 20$$

Multiplying by $(s+5)^3$and solving for $s = -5$;

$$\frac{100(s+25)}{s} = \frac{K_1(s+5)^3}{s} + K_2 + K_3(s+5) + K_4(s+5)^2$$
$$\frac{100(20)}{-5} = K_2 = -400$$

To solve for $(s+5)^2$we need to differentiate and solve for $s = -5$

$$\frac{d}{ds}\left[\frac{100(s+25)}{s}\right] = \frac{d}{ds}\left[\frac{K_1(s+5)^3}{s} + K_2 + K_3(s+5) + K_4(s+5)^2\right]$$
$$100\left[\frac{s-(s+25)}{s^2}\right] = K_3 = \frac{100(-25)}{(-5)^2} = -100$$

To solve for $s+5$ we need to differentiate again and solve for $s = -5$

$$\frac{d^2}{ds}\left[\frac{100(s+25)}{s}\right] = \frac{d^2}{ds}\left[\frac{K_1(s+5)^3}{s} + K_2 + K_3(s+5) + K_4(s+5)^2\right]$$

$$100\left(\frac{25}{2s^3}\right) = 2K_4 \rightarrow\ K_4 = \frac{100}{2}\left(\frac{25}{2(-5)^3}\right) = -20$$

Thus

$$F(s) = \frac{100(s+25)}{s(s+5)^3} = \frac{20}{s} - \frac{400}{(s+5)^3} - \frac{100}{(s+5)^2} - \frac{20}{s+5}$$

Once again test for validity and take the inverse if valid

$$\mathcal{L}^{-1}\left\{\frac{100(s+25)}{s(s+5)^3}\right\} = (20 - 200t^2e^{-5t} - 100te^{-5t} - 20e^{-5t})u(t)$$

Partial Fraction Expansion: Repeated Complex Roots of D(s)
Same as the repeated real roots but with complex numbers.

$$Example\ \ F(s) = \frac{768}{(s^2+6s+25)^2} = \frac{768}{(s+3-j4)^2(s+3+j4)^2}$$

Expanding

$$\frac{768}{(s+3-j4)^2(s+3+j4)^2} = \frac{K_1}{(s+3-j4)^2} + \frac{K_2}{s+3-j4} + \frac{K_1^*}{(s+3+j4)^2} + \frac{K_2^*}{s+3+j4}$$

Need to evaluate for $K_1 \& K_2$

$$For\ s = -3 + j4: \qquad K_1 = \frac{768}{(s+3+j4)^2} = \frac{768}{(j8)^2} = -12 \quad \rightarrow \quad K_1^* = -12$$

$$Taking\ the\ derivative\ \ and\ solving\ for\ s = -3 + j4:$$

$$K_2 = \frac{d}{ds}\left[\frac{768}{(s+3+j4)^2}\right] = -\frac{2(768)}{(s+3+j4)^3} = -\frac{2(768)}{(j8)^3} = -j3 = 3\angle -90°$$

$$K_2^* = j3 = 3\angle 90°$$

Rewriting

$$F(s) = \left[\frac{-12}{(s+3-j4)^2} + \frac{-12}{s+3-j4}\right] + \left(\frac{3\angle -90°}{(s+3+j4)^2} + \frac{3\angle 90°}{s+3+j4}\right)$$

Taking the inverse transform

$$f(t) = [-24te^{-3t}cos4t + 6e^{-3t}\cos(4t - 90°)]u(t)$$

Note:
If $F(s)$ has a real root a of multiplicity r in the denominator the expansion looks like:

$$\frac{K}{(s+a)^r}\ \ with\ the\ inverse\ \frac{Kt^{r-1}e^{-at}}{(r-1)!}u(t)$$

If $F(s)$ has a complex root a of multiplicity r in the denominator the expansion looks like:

$$\frac{K}{(s+\alpha-j\beta)^r} + \frac{K^*}{(s+\alpha+j\beta)^r}$$

$with\ the\ inverse$

$$\left[\frac{2|K|t^{r-1}}{(r-1)!}e^{-\propto t}\cos(\beta t + \theta)\right]u(t)$$

Partial Fraction Expansion: Improper Rational Functions

An improper rational function can be expanded into a polynomial plus a proper rational function. The polynomial is then transformed into impulse functions and derivatives of impulse functions. The proper rational function transformed as before.

$$Example \qquad F(s) = \frac{s^4 + 13s^3 + 66s^2 + 200s + 300}{s^2 + 9s + 20}$$

Divide the denominator into the numerator until the remainder is a proper rational function.

$$F(s) = s^2 + 4s + 10 + \frac{30s + 100}{s^2 + 9s + 20}$$

Expand the proper rational function into a sum of partial fractions:

$$\frac{30s+100}{s^2+9s+20} = \frac{30s+100}{(s+4)(s+5)} = \frac{-20}{s+4} + \frac{50}{s+5}$$

Substitute the partial fraction back into the equation

$$F(s) = s^2 + 4s + 10 - \frac{20}{s+4} + \frac{50}{s+5}$$

Now find the inverse transform

$$f(t) = \frac{d^2\delta(t)}{dt^2} + 4\frac{d\delta(t)}{dt} + 10\delta(t) - (20e^{-4t} - 50e^{-5t})u(t)$$

12.8 Poles and Zeros of F(s)

A rational function can be expressed as a ratio of two factored polynomials:

$$F(s) = \frac{N(s)}{D(s)} = \frac{K(s+z_1)(s+z_2)\cdots(s+z_n)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

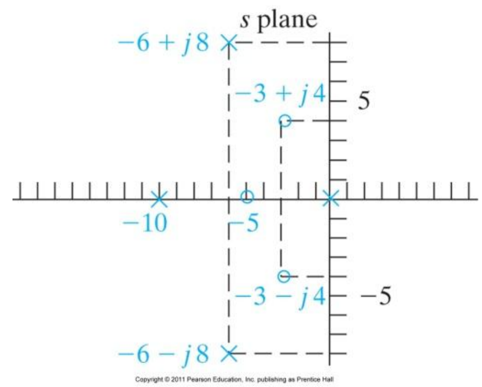
Where K is the constant $\frac{a_n}{b_m}$

The **poles of F(s)** are the roots of the denominator

The **zeros of F(s)** are the roots of the numerator

Plotting the poles and zeros of F(s) may be useful.

$$ex.\ F(s) = \frac{10(s+5)(s+3-j4)(s+3+j4)}{s(s+10)(s+6-j8)(s+6+j8)}$$



When plotting in the s-plan the x-axis is the real values and y-axis the imaginary

X's are used to represent poles

$0's$ are used to represent the zeros

Note: for the text we look at the pole and zeros in a finite plane.

12.9 Initial- and Final Value Theorems

Let us use F(s) to determine the behavior of f (t) at 0 and ∞. We can check the initial and final values of f (t) to see if it conforms with the known behavior before we take the inverse transform

$$initial - value\ theorem:\ \lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$$

$$final - value\ theorem:\ \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Assumptions

Initial-value theorem

i. f (t) has no impulse functions
ii. Only valid if poles of F(s) are in the left half of the s-plane (except for a first order pole at the origin)

Final-value theorem

i. Only valid if poles of F(s) are in the left half of the s-plane (except for a first order pole at the origin)

(Can review proof in the book)

The Application of Initial- and Final-Value Theorems

$$For \qquad F(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)}$$

$$and \qquad f(t) = \left(-12e^{-6t} + 10e^{-j53.13^\circ}e^{-(3-j4)t} + 10e^{j53.13^\circ}e^{-(3+j4)t}\right)u(t)$$

$initial - value\ theorem$:

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{100s(s+3)}{(s+6)(s^2+6s+25)} = \lim_{s\to\infty} \frac{100s^2\left(1+\frac{3}{s}\right)}{s^3\left(1+\frac{6}{s}\right)\left(1+\frac{6}{s}+\frac{25}{s^2}\right)} = 0$$

$$\lim_{t\to 0^+} f(t) = \lim_{t\to 0^+} \left(-12e^{-6t} + 10e^{-j53.13^\circ}e^{-(3-j4)t} + 10e^{j53.13^\circ}e^{-(3+j4)t}\right)u(t)$$
$$= [-12 + 20\cos(-53.13^\circ)](1) = 0$$

$final - value\ theorem$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{100s(s+3)}{(s+6)(s^2+6s+25)} = 0$$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}\left(-12e^{-6t} + 10e^{-j53.13^\circ}e^{-(3-j4)t} + 10e^{j53.13^\circ}e^{-(3+j4)t}\right)u(t) = 0$$

Therefore the initial and final values for this expression are 0